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# Shallow-water theory for arbitrary slopes of the bottom

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A new shallow-water theory valid for arbitrary bottom slope, due to Bouchut *et al.* (2003), is derived systematically by a scaling method. The fact that the pressure is hydrostatic, and the form of the velocity parallel to the bottom, are consequences of the scaling method, and need not be assumed.

## 1. Introduction

The usual shallow-water theory is a simplified form of the equations of motion of a shallow layer of fluid over a bottom which is horizontal, or has a small slope. (Stoker 1957, pp. 22–26). To treat avalanches, Savage & Hutter (1991) modified this theory to allow large, but slowly varying, bottom slopes. Bouchut *et al.* (2003) recently devised a new shallow-water theory which permits arbitrary bottom slopes and slope variations. To do so they assumed that the pressure is hydrostatic, and that the velocity parallel to the bottom has a particular variation in the normal direction.

We shall derive the theory of Bouchut *et al.* (2003) systematically, without making these assumptions. Instead we scale each variable by a suitable power of a small parameter $\varepsilon$. When we omit the highest powers of $\varepsilon$ from the scaled equations, we obtain the desired shallow-water theory.

The powers of $\varepsilon$ we use differ from those in Friedrichs' derivation of the usual shallow-water theory (Stoker 1957, pp. 26–30). Keller & Weitz (1957) used yet another scaling to derive a hydraulic theory for steep bottom slopes, and Geer & Keller (1979) extended that theory to a full asymptotic expansion.

The scaling and the scaled equations are introduced in § 2. In § 3 the new shallow-water theory is derived, and the resulting equations are examined and simplified. In the Appendix a method of deriving the scaling is presented.

## 2. Formulation

We begin with the Euler equations of motion for an incompressible fluid, written in curvilinear coordinates $x$, $y$, with a curve $C$ as the coordinate line $y = 0$. If the bottom is smooth and not moving, we follow Bouchut *et al.* (2003) and choose it as $C$; otherwise we choose $C$ to be a fixed smooth curve near the bottom. Distance along $C$ is $x$ and distance from $C$ along the normal is $y$. The $x$- and $y$-components of velocity are $u(x, y, t)$ and $v(x, y, t)$, and the pressure divided by the density is $p(x, y, t)$. The angle between $C$ and the horizontal is $\theta(x)$, the given bottom surface is $y = -h(x, t)$, the free top surface is $y = \eta(x, t)$, and $g$ is the acceleration due to gravity.

We introduce new primed variables by rescaling the original ones with powers of a small parameter $\varepsilon$, which is the ratio of a typical dimension of the flow in the $y$-direction to a typical dimension in the $x$-direction. Thus $x = x'$, $y = \varepsilon y'$, $\eta = \varepsilon\eta'$, and $h\,l = \varepsilon h'$ in view of the meaning of $\varepsilon$. The other variables are scaled as follows:

$$u = u', \quad v = \varepsilon v', \quad p = \varepsilon p', \quad t = t'. \tag{2.1}$$

In terms of the new variables we write the continuity equation, the equations of motion, the condition of irrotationality, the kinematic and dynamic conditions at the free surface, and the kinematic condition at the bottom as follows, with the primes omitted, and with $J(x, y) = 1 - \varepsilon y \theta_x(x)$:

$$u_x + (Jv)_y = 0, \tag{2.2}$$

$$Ju_t + uu_x + Jvu_y + \varepsilon p_x = -Jg\sin\theta + \varepsilon uv\theta_x, \tag{2.3}$$

$$\varepsilon^2(Jv_t + uv_x + Jvv_y) + \varepsilon p_y = -\varepsilon Jg\cos\theta - \varepsilon u^2\theta_x, \tag{2.4}$$

$$\varepsilon^2 v_x - (Ju)_y = 0, \tag{2.5}$$

$$\eta_t + J^{-1}u\eta_x - v = 0 \qquad \text{at} \qquad y = \eta(x, t), \tag{2.6}$$

$$p = 0 \qquad \text{at} \qquad y = \eta(x, t), \tag{2.7}$$

$$h_t + J^{-1}uh_x + v = 0 \qquad \text{at} \qquad y = -h(x, t). \tag{2.8}$$

We divided by $\varepsilon$ to get (2.5)–(2.8) and multiplied by $\varepsilon$ to get (2.4).

To obtain an asymptotic theory, we omit the terms of order $\varepsilon^2$ from (2.4) and (2.5). Then (2.4) becomes, after cancelling the common factor $\varepsilon$,

$$p_y = -Jg\cos\theta - u^2\theta_x. \tag{2.9}$$

With the $\varepsilon^2$ term omitted, (2.5) becomes

$$(Ju)_y = 0. \tag{2.10}$$

Equation (2.9) is the hydrostatic equation for $p$, with the inclusion of a centrifugal force term. Equation (2.10) states that $Ju$ is independent of $y$. These are the two conditions which were assumed by Bouchut *et al.* (2003) in their derivation. Here they arise automatically as a consequence of the scaling.

## 3. The shallow-water theory

Now we shall derive the new shallow-water theory from (2.2), (2.3) and (2.6)–(2.10). From (2.10) and the definition of $J$ we get

$$u(x, y, t) = U(x, t)\,[1 - \varepsilon y\theta_x(x)]^{-1}. \tag{3.1}$$

By using (3.1) in (2.9), integrating with respect to $y$, and using (2.7) we get

$$p(x, y, t) = (\eta - y)g\cos\theta + \frac{U^2(x, t)}{2}\{[1 - \eta(x, t)\theta_x(x)]^{-1} - [1 - y\theta_x(x)]^{-2}\}. \tag{3.2}$$

These two equations become exactly (3.9) of Bouchut *et al.* (2003) when we set $\varepsilon = 1$, which is equivalent to reintroducing the original variables.

We use (3.1) for $u$ in (2.2) and integrate with respect to $y$ from $y = -h$, using the boundary condition (2.8). This yields the following expression for $v$:

$$v(x, y, t) = -J^{-1}(x, y)\partial_x \int_{-h}^{y} U[1 - \varepsilon y\theta_x]^{-1}\,\mathrm{d}y - J^{-1}(x, y)J(x, -h)h_t$$
$$= J^{-1}(x, y)\partial_x \left[\frac{U}{\varepsilon\theta_x}\log\frac{1 - \varepsilon y\theta_x}{1 + \varepsilon h\theta_x}\right] - J^{-1}(x, y)J(x, -h)h_t. \quad (3.3)$$

Upon using (3.3) for $v$ in (2.6) we get the mass conservation equation

$$J(x, \eta)\eta_t + J(x, -h)h_t + \partial_x \left[\frac{U}{\varepsilon\theta_x}\log\frac{1 - \varepsilon\eta\theta_x}{1 + \varepsilon h\theta_x}\right] = 0. \quad (3.4)$$

Next we use (3.1) for $u$ and (3.2) for $p$ in (2.3). After some cancellation we get

$$U_t + \partial_x \left[\frac{U^2}{2J^2(x, \eta)} + \eta g\cos\theta\right] = -g\sin\theta. \quad (3.5)$$

The equations (3.4) and (3.5) for $\eta(x, t)$ and $U(x, t)$ constitute the shallow-water theory for bottoms of any slope. The functions $\theta(x)$ and $h(x, t)$ are assumed to be known. In terms of $\eta$ and $U$, the quantities $u$, $p$ and $v$ are given by (3.1)–(3.3).

When $h(x, t) \equiv 0$ and $\varepsilon = 1$, (3.4) and (3.5) become equations (3.1) of Bouchut *et al.* (2003). Thus we have succeeded in deriving the new shallow-water theory of Bouchut *et al.* (2003) just by scaling the variables appropriately in powers of $\varepsilon$, and omitting terms of order $\varepsilon^2$. The fact that the pressure is hydrostatic, and that $u$ has the form (3.1), are consequences of the scaling, and need not be assumed. Higher-order terms in the expansion of the solution with respect to $\varepsilon$ could be obtained by using the scaled equations. Other problems outside the scope of the usual shallow-water theory can now be formulated and solved by means of the new theory.

In the Appendix we describe the method we used to determine the scaling (2.1), which was used in our derivation. This method can be used in other problems when some knowledge about the form of the desired asymptotic theory is available.

## Appendix. Determining the scaling

To find the scaling (2.1) we begin with $x = x'$, $y = \varepsilon y'$, $\eta = \varepsilon\eta'$, $h = \varepsilon h'$ and the general scaling

$$u = \varepsilon^a u', \quad v = \varepsilon^b v', \quad p = \varepsilon^c p', \quad t = \varepsilon^d t'. \quad (\mathrm{A}\,1)$$

We use the new variables in the original equations and omit the primes to obtain

$$\varepsilon^a u_x + \varepsilon^{b-1}(Jv)_y = 0, \quad (\mathrm{A}\,2)$$

$$\varepsilon^{a-d}Ju_t + \varepsilon^{2a}uu_x + \varepsilon^{a+b-1}Jvu_y + \varepsilon^c p_x = -Jg\sin\theta + \varepsilon^{a+b}uv\theta_x, \quad (\mathrm{A}\,3)$$

$$\varepsilon^{b-d}Jv_t + \varepsilon^{a+b}uv_x + \varepsilon^{2b-1}Jvv_y + \varepsilon^{c-1}p_y = -Jg\cos\theta - \varepsilon^{2a}u^2\theta_x, \quad (\mathrm{A}\,4)$$

$$\varepsilon^b v_x - \varepsilon^{a-1}(Ju)_y = 0, \quad (\mathrm{A}\,5)$$

$$\varepsilon^{1-d}\eta_t + \varepsilon^{a+1}J^{-1}u\eta_x - \varepsilon^b v = 0 \quad \text{at} \quad y = \eta(x, t), \quad (\mathrm{A}\,6)$$

$$\varepsilon^c p = 0 \quad \text{at} \quad y = \eta(x, t), \quad (\mathrm{A}\,7)$$

$$\varepsilon^{1-d}h_t + \varepsilon^{a+1}J^{-1}uh_x + \varepsilon^b v = 0 \quad \text{at} \quad y = -h(x, t). \quad (\mathrm{A}\,8)$$

Here $J = 1 - \varepsilon y\theta_x$.

Now we shall determine the exponents $a$, $b$, $c$ and $d$ so that (A 2)–(A 8) yield the shallow-water theory appropriate to finite or steep slopes for $\varepsilon \ll 1$. Thus we require that both terms in (A 2) remain as $\varepsilon \to 0$, so we must have

$$a = b - 1. \tag{A 9}$$

In (A 3) the terms $Ju_t$, $uu_x$ and $-Jg\sin\theta$ should remain as $\varepsilon \to 0$, so we must have

$$a - d = 0, \quad 2a = 0, \quad c \geqslant 0, \quad a + b - 1 \geqslant 0, \quad a + b \geqslant 0. \tag{A 10}$$

In (A 4) $p_y$ and $-Jg\cos\theta$ should remain while $Jv_t$, $uv_x$, and $Jvv_y$ should not remain, so we require

$$c - 1 = 0, \quad b - d > 0, \quad a + b > 0, \quad 2b - 1 > 0, \quad 2a \geqslant 0. \tag{A 11}$$

In (A 5) we want $(Ju)_y$ to remain, but not $v_x$, so we need

$$a - 1 < b. \tag{A 12}$$

In (A 6) and (A 8) all terms should remain, so we must have

$$1 - d = a + 1 = b. \tag{A 13}$$

Equations (A 9)–(A 13) contain six homogeneous linear equations and eight inequalities for the four exponents. They have the unique solution

$$a = 0, \quad b = 1, \quad c = 1, \quad d = 0. \tag{A 14}$$

Using these exponents in (A 1) yields the scaling (2.1).

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